Arc Length

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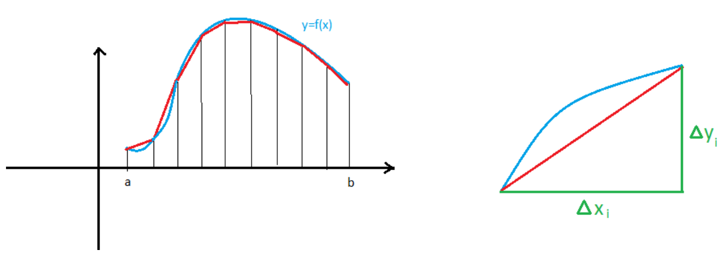
We are going to find a way to compute the length of a curve that is the graph of some function $y = f(x)$ using the idea of Riemann sum and hence integration.

Given a function $y = f(x)$ on an interval $[a, b]$, we call its graph the curve C. In order to compute the length of C, we first divide the interval $[a, b]$ into n pieces with endpoints $a = x_0 < x_1 < ..., x_{n-1} < x_n = b$, and denote $\Delta x_i = x_i - x_{i-1}$, denote $y_i = f(x_i)$, $\Delta y_i = y_i - y_{i-1}$. We connect the line segment between $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$.

The Mean Value Theorem tells us there exists $x_i^* \in (x_{i-1}, x_i)$ such that $f'(x_i^*) = \frac{\Delta y_i}{\Delta x_i}$. The length of this segment is computed by Pythagorean Theorem:

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + (\frac{\Delta y_i}{\Delta x_i})^2}\Delta x_i = \sqrt{1 + (f'(x_i^*))^2}\Delta x_i$$



Sum up the length of these line segments:

$$\sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x_i$$

As $\max \Delta x_i \to 0$, this summation will converge to the length of the curve C:

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\Delta x_i = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

So we define the **arc length** of the graph of $y = f(x)$ between $(a, f(a))$ and $(b, f(b))$ to be:

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

If we fix an initial point $(a, f(a))$, then for each $x \in [a, b]$, we can get an arc length for the part of graph on $[a, x]$, hence we get an arc length function:

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$$

Example 1. *Find the length of the graph of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.*

Between these two points, the curve is above x-axis, so $y = x^{\frac{3}{2}}$. So the arc length is

$$\int_1^4 \sqrt{1 + [(x^{\frac{3}{2}})']^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{80\sqrt{10} - 13\sqrt{13}}{27}$$

Example 2. *Find the length of the arc of the parabola $x = y^2$ from $(1, -1)$ to $(1, 1)$.*

In this example, x is a function of y, so we need to apply the formula for arc length the other way round.

$$\begin{aligned}
\int_{-1}^1 \sqrt{1 + (\frac{dx}{dy})^2}\, dy = \int_{-1}^1 \sqrt{1 + 4y^2}\, dy = \int_{-1}^1 \sqrt{1 + 4y^2}\, dy &= \int_{-\tan^{-1} 2}^{\tan^{-1} 2} \sec t\, d \tan t \\
&= \int_{-\tan^{-1} 2}^{\tan^{-1} 2} \frac{1}{\cos^3 t}\, dt \\
&= \int_{-\tan^{-1} 2}^{\tan^{-1} 2} \frac{1}{\cos^4 t}\, d \sin t \\
&= \int_{-\tan^{-1} 2}^{\tan^{-1} 2} \frac{1}{(1 - \sin^2 t)^2}\, d \sin t \\
&= \int_{-\frac{2}{\sqrt{5}}}^{\frac{2}{\sqrt{5}}} \frac{1}{(1 - u^2)^2}\, du \\
&= \frac{\ln(2 + \sqrt{5})}{2} + \sqrt{5}
\end{aligned}$$

Example 3. *Find the arc length function for the curve* $y = x^2 - \frac{1}{8}\ln x$ *taking* $(1,1)$ *as the starting point.*

$$\begin{aligned}
s(x) &= \int_1^x \sqrt{1 + [(t^2 - \frac{1}{8}\ln t)']^2}\,dx \\
&= \int_1^x \sqrt{1 + (2 - \frac{1}{8t})^2}\,dx \\
&= \int_1^x 2t + \frac{1}{8t}\,dt \\
&= x^2 + \frac{1}{8}\ln x - 1
\end{aligned}$$